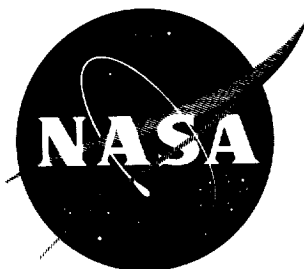


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# TECHNICAL NOTE

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## SATELLITE MOTION IN THE VICINITY OF CRITICAL INCLINATION

David Fisher

Goddard Space Flight Center

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION  
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by  
David Fisher  
*Goddard Space Flight Center*

## SUMMARY

A development of the motion of a close earth satellite in the vicinity of critical inclination is given to order  $k_2$ . The method followed is that of expanding the potential in powers of  $G$  in the neighborhood of  $G_0$ . The odd harmonic of the potential function is included in this expansion. The equations of motion are then given in the form suitable for numerical computation by electronic computers. It is shown that this approximation is analogous to the motion of a pendulum.



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## INTRODUCTION

Since the quantity  $5 \cos^2 I - 1$ , where  $I$  is the inclination of the orbit plane, appears as a divisor in the solutions of the equations governing the motion of close earth satellites, the motion of a satellite in the vicinity of  $\cos^2 I = 1/5$  is a subject of much interest (References 1-4). Analytic studies of the motion of a close earth satellite to the order  $\sqrt{k_2}$  in the vicinity of  $\cos^2 I = 1/5$  (called the critical inclination) have shown that this satellite motion is exactly analogous to the motion of a pendulum free to oscillate about a horizontal axis. The motion of this type of pendulum has been studied extensively (References 5 and 6); and consequently the motion of a close earth satellite in the vicinity of critical inclination is now well understood.

For the prediction of the motion of close earth satellites, however, it is readily possible to increase the accuracy of the solutions by numerical methods with the use of modern electronic computers, as suggested by Bailie, and by Hori (Reference 3). Accordingly, a numerical approach suggested by the works of Hagihara (Reference 1) and of Garfinkel (Reference 4) has been adopted in the development discussed herein.

## EXPANSION OF THE HAMILTONIAN

The Delaunay variables are defined by the following expressions:

$$L = (\mu a)^{1/2},$$

$$G = L(1 - e^2)^{1/2},$$

$$H = G \cos I,$$

$$l = \text{mean anomaly},$$

$g$  = argument of perigee,

$h$  = argument of the ascending node.

It is assumed that  $L$  and  $H$  are known constants.  $G$  is always taken as positive; consequently when  $\cos I$  is negative,  $H$  is also negative.

Brouwer's Hamiltonian (Reference 7) to order  $k_2^2$  is given by

$$F = F_0 + F_1 + F_{2s} + Q_2 \cos 2g + Q_3 \sin g. \quad (1)$$

By converting Equation 1 to Vanguard units (i.e.,  $k_4/k_2^2 = 11/5$ ,  $A_{3,0} = -2k_3$ , and  $\mu = 1$ ), we have

$$F_0 = \frac{1}{2L^2},$$

$$F_1 = \frac{-k_2}{2L^3G^3} \left( 1 - 3 \frac{H^2}{G^2} \right),$$

$$F_{2s} = \frac{k_2^2}{160L^5G^5} \left[ -123 + 1710 \frac{H^2}{G^2} - 2235 \frac{H^4}{G^4} \right. \\ \left. + 60 \frac{L}{G} \left( 1 - 6 \frac{H^2}{G^2} + 9 \frac{H^4}{G^4} \right) + 5 \frac{L^2}{G^2} \left( 51 - 630 \frac{H^2}{G^2} + 875 \frac{H^4}{G^4} \right) \right]$$

$$Q_2 = \frac{k_2^2}{L^5G^5} \left( 1 - \frac{L^2}{G^2} \right) \left( \frac{19}{16} - 8 \frac{H^2}{G^2} + \frac{109}{16} \frac{H^4}{G^4} \right)$$

$$Q_3 = - \frac{3k_3}{4L^3G^5} e \sin I \left( 1 - 5 \frac{H^2}{G^2} \right),$$

where

$$e = \left( 1 - \frac{G^2}{L^2} \right)^{1/2},$$

$$\sin I = \left( 1 - \frac{H^2}{G^2} \right)^{1/2}.$$

In order to study the motion in the vicinity of critical inclination, it is convenient to expand the Hamiltonian, as was done by Garfinkel (Reference 4) in powers of  $\delta = G - G_0$ , where  $G_0$  is the value of  $G$  at the epoch. Then

$$\begin{aligned}
 F = & F_0 + (F_1)_0 + (F_{2s})_0 + (Q_2)_0 \cos 2g + (Q_3)_0 \sin g \\
 & + \left[ \left( \frac{\partial F_1}{\partial G} \right)_0 + \left( \frac{\partial F_{2s}}{\partial G} \right)_0 + \left( \frac{\partial Q_2}{\partial G} \right)_0 \cos 2g + \left( \frac{\partial Q_3}{\partial G} \right)_0 \sin g \right] \delta \\
 & + \frac{1}{2} \left[ \left( \frac{\partial^2 F_1}{\partial G^2} \right)_0 + \left( \frac{\partial^2 F_{2s}}{\partial G^2} \right)_0 + \left( \frac{\partial^2 Q_2}{\partial G^2} \right)_0 \cos 2g + \left( \frac{\partial^2 Q_3}{\partial G^2} \right)_0 \sin g \right] \delta^2 \\
 & + \frac{1}{6} \left( \frac{\partial^3 F_1}{\partial G^3} \right)_0 \delta^3 + \frac{1}{24} \left( \frac{\partial^4 F_1}{\partial G^4} \right)_0 \delta^4 .
 \end{aligned} \tag{2}$$

The values of the coefficients of  $\delta$  are:

$$\left( \frac{\partial F_1}{\partial G} \right)_0 = \frac{3k_2}{2L^3 G_0^4} (1 - 5 \cos^2 I_0) ,$$

$$\left( \frac{\partial^2 F_1}{\partial G^2} \right)_0 = \frac{3k_2}{L^3 G_0^5} (-2 + 15 \cos^2 I_0) , \tag{3a}$$

$$\left( \frac{\partial^3 F_1}{\partial G^3} \right)_0 = \frac{15k_2}{L^3 G_0^6} (2 - 21 \cos^2 I_0) , \tag{3b}$$

$$\left( \frac{\partial^4 F_1}{\partial G^4} \right)_0 = \frac{180k_2}{L^3 G_0^7} (-1 + 14 \cos^2 I_0) , \tag{3c}$$

$$\begin{aligned}
 \left( \frac{\partial F_{2s}}{\partial G} \right)_0 = & \frac{k_2^2}{L^5 G_0^6} \left[ \frac{123}{32} - \frac{1197}{16} \cos^2 I_0 + \frac{4023}{32} \cos^4 I_0 - \frac{L}{G_0} \left( \frac{9}{4} - 18 \cos^2 I_0 + \frac{135}{4} \cos^4 I_0 \right) \right. \\
 & \left. - \frac{L^2}{G_0^2} \left( \frac{357}{32} - \frac{2835}{16} \cos^2 I_0 + \frac{9625}{32} \cos^4 I_0 \right) \right] , \tag{3d}
 \end{aligned}$$

$$\begin{aligned} \left( \frac{\partial^2 F_{2s}}{\partial G^2} \right)_0 &= \frac{k_2^2}{L^5 G_0^7} \left[ -\frac{369}{16} + \frac{1197}{2} \cos^2 I_0 - \frac{20115}{16} \cos^4 I_0 + \frac{L}{G_0} \left( \frac{63}{4} - 162 \cos^2 I_0 + \frac{1485}{4} \cos^4 I_0 \right) \right. \\ &\quad \left. + \frac{L^2}{G_0^2} \left( \frac{357}{4} - \frac{14175}{8} \cos^2 I_0 + \frac{28875}{8} \cos^4 I_0 \right) \right] , \quad (3e) \end{aligned}$$

$$\begin{aligned} \left( \frac{\partial Q_2}{\partial G} \right)_0 &= \frac{k_2^2}{L^5 G_0^6} \left[ -\frac{95}{16} + 56 \cos^2 I_0 - \frac{981}{16} \cos^4 I_0 \right. \\ &\quad \left. + \frac{L^2}{G_0^2} \left( \frac{133}{16} - 72 \cos^2 I_0 + \frac{1199}{16} \cos^4 I_0 \right) \right] , \quad (3f) \end{aligned}$$

$$\begin{aligned} \left( \frac{\partial^2 Q_2}{\partial G^2} \right)_0 &= \frac{k_2^2}{L^5 G_0^7} \left[ \frac{285}{8} - 448 \cos^2 I_0 + \frac{4905}{8} \cos^4 I_0 \right. \\ &\quad \left. - \frac{L^2}{G_0^2} \left( \frac{133}{2} - 720 \cos^2 I_0 + \frac{3597}{4} \cos^4 I_0 \right) \right] , \quad (3g) \end{aligned}$$

$$\left( \frac{\partial Q_3}{\partial G} \right)_0 = \frac{3k_3}{4L^3 G_0^6} \left[ 5e_0 \sin I_0 (1 - 7 \cos^2 I_0) + (1 - 5 \cos^2 I_0) \left( \frac{\sin I_0}{e_0} - \frac{e_0}{\sin I_0} \right) \right] \quad (3h)$$

$$\begin{aligned} \left( \frac{\partial^2 Q_3}{\partial G^2} \right)_0 &= -\frac{3k_3}{4L^3 G_0^7} \left[ (30 - 280 \cos^2 I_0) e_0 \sin I_0 + (11 - 75 \cos^2 I_0) \left( \frac{\sin I_0}{e_0} - \frac{e_0}{\sin I_0} \right) \right. \\ &\quad \left. + (5 \cos^2 I_0 - 1) \left( \frac{\sin I_0}{e_0} + \frac{e_0}{\sin I_0} \right) \left( \cot^2 I_0 + \frac{G_0^2}{L^2 e_0^2} \right) \right] . \quad (3i) \end{aligned}$$

An integral of the motion is given by the equation

$$F = C . \quad (4)$$

To evaluate C we set  $G = G_0$ , or  $\delta = 0$ , and  $g = g_0$  to obtain

$$C = F_0 + (F_1)_0 + (F_{2s})_0 + (Q_2)_0 \cos 2g_0 + (Q_3)_0 \sin g_0 , \quad (5)$$

so that Equation 4 yields the following relation between  $\delta$  and  $g$ :

$$\begin{aligned}
 & (Q_2)_0 [\cos 2g - \cos 2g_0] + (Q_3)_0 [\sin g - \sin g_0] \\
 & + \left[ \left( \frac{\partial F_1}{\partial G} \right)_0 + \left( \frac{\partial F_{2s}}{\partial G} \right)_0 + \left( \frac{\partial Q_2}{\partial G} \right)_0 \cos 2g + \left( \frac{\partial Q_3}{\partial G} \right)_0 \sin g \right] \delta \\
 & + \frac{1}{2} \left[ \left( \frac{\partial^2 F_1}{\partial G^2} \right)_0 + \left( \frac{\partial^2 F_{2s}}{\partial G^2} \right)_0 + \left( \frac{\partial^2 Q_2}{\partial G^2} \right)_0 \cos 2g + \left( \frac{\partial^2 Q_3}{\partial G^2} \right)_0 \sin g \right] \delta^2 \\
 & + \frac{1}{6} \left( \frac{\partial^3 F_1}{\partial G^3} \right)_0 \delta^3 + \frac{1}{24} \left( \frac{\partial^4 F_1}{\partial G^4} \right)_0 \delta^4 = 0.
 \end{aligned} \tag{6}$$

## THE MEAN MOTION OF THE ANGULAR VARIABLES

The equations of motion are given by

$$\left. \begin{aligned}
 \frac{dL}{dt} &= 0, & \frac{dl}{dt} &= -\frac{\partial F}{\partial L}, \\
 \frac{dG}{dt} &= \frac{\partial F}{\partial g}, & \frac{dg}{dt} &= -\frac{\partial F}{\partial G}, \\
 \frac{dH}{dt} &= 0, & \frac{dh}{dt} &= -\frac{\partial F}{\partial H}.
 \end{aligned} \right\} \tag{7}$$

In order to find the mean motion of the angular variables, we set the Hamiltonian

$$F = F^* + 2(Q_2)_0 \cos^2 g - (Q_3)_0 \sin^2 \frac{g}{2}, \tag{8}$$

where

$$F^* = F_0 + F_1 + F_{2s} - Q_2 + Q_3. \tag{9}$$

### The Mean Motion of $l$

The mean motion of  $l$  is given by

$$\begin{aligned}
\frac{dl'}{dt} &= - \frac{\partial F^*}{\partial L} \\
&= \frac{1}{L^3} - \frac{3k_2}{2L^4 G_0^3} (1 - 3 \cos^2 I_0) \\
&\quad + \frac{k_2^2}{160 L^6 G_0^5} \left[ 5 (-123 + 1710 \cos^2 I_0 - 2235 \cos^4 I_0) \right. \\
&\quad \left. + 240 \frac{L}{G_0} (1 - 6 \cos^2 I_0 + 9 \cos^4 I_0) \right. \\
&\quad \left. + 15 \frac{L^2}{G_0^2} (51 - 630 \cos^2 I_0 + 875 \cos^4 I_0) \right] \\
&\quad - \frac{k_2^2}{L^6 G_0^5} \left( 5 - 3 \frac{L^2}{G_0^2} \right) \left( \frac{19}{16} - 8 \cos^2 I_0 + \frac{109}{16} \cos^4 I_0 \right) \\
&\quad + \frac{3k_3}{4L^4 G_0^5} \left( \frac{1}{e_0} \frac{G_0^2}{L^2} - 3e_0 \right) \sin I_0 (1 - 5 \cos^2 I_0) .
\end{aligned} \tag{10}$$

### The Mean Motion of $h$

The mean motion of  $h$  is given by

$$\begin{aligned}
\frac{dh'}{dt} &= - \frac{3k_2 \cos I_0}{L^3 G_0^4} - \frac{k_2^2 \cos I_0}{160 L^5 G_0^6} \left[ 3420 - 8940 \cos^2 I_0 \right. \\
&\quad \left. + 60 \frac{L}{G_0} (-12 + 36 \cos^2 I_0) + 5 \frac{L^2}{G_0^2} (-1260 + 3500 \cos^2 I_0) \right] \\
&\quad - \frac{k_2^2}{L^5 G_0^6} \cos I_0 \left( 1 - \frac{L^2}{G_0^2} \right) \left( -16 + \frac{109}{4} \cos^2 I_0 \right) \\
&\quad - \frac{3k_3}{4L^3 G_0^6} e_0 \cos I_0 \left[ \frac{(1 - 5 \cos^2 I_0)}{\sin I_0} + 10 \sin I_0 \right] .
\end{aligned} \tag{11}$$

### The Mean Motion of g

In close proximity to the critical inclination, (i.e., when  $|1 - 5 \cos^2 I| = \lambda \sqrt{k_2}$  where  $\lambda < 1$ ), the mean motion of g may be zero. This is the case of libration and will be discussed later. When the mean motion of g is not zero, it is given by

$$\begin{aligned} \frac{dg'}{dt} = & -\frac{3 k_2}{2 L^3 G_0^4} (1 - 5 \cos^2 I_0) \\ & + \frac{k_2^2}{32 L^5 G_0^6} \left[ -313 + 4186 \cos^2 I_0 - 5985 \cos^4 I_0 \right. \\ & \left. + \frac{L}{G_0} (72 - 576 \cos^2 I_0 + 1080 \cos^4 I_0) + \frac{L^2}{G_0^2} (623 - 7974 \cos^2 I_0 + 12023 \cos^4 I_0) \right] \\ & - \frac{3 k_3}{4 L^3 G_0^6} \left[ 5 e_0 \sin I_0 (1 - 7 \cos^2 I_0) + (1 - 5 \cos^2 I_0) \left( \frac{\sin I_0}{e_0} - \frac{e_0}{\sin I_0} \right) \right]. \end{aligned} \quad (12)$$

## THE SOLUTION INCLUDING THE LONG PERIOD TERMS

The motions of l, g, and h are given by the canonical equations:

$$\left. \begin{aligned} \dot{l} &= -\frac{\partial F}{\partial L}, \\ \dot{g} &= -\frac{\partial F}{\partial G}, \\ \dot{h} &= -\frac{\partial F}{\partial H}. \end{aligned} \right\} \quad (13)$$

### The Motion of g

Since

$$\dot{g} = -\frac{\partial F}{\partial G} = -\frac{\partial F}{\partial \delta}; \quad (14)$$

and from Equation 2 we arrive at

$$\begin{aligned}
 \dot{g} = & \left[ \left( \frac{\partial F_1}{\partial G} \right)_0 + \left( \frac{\partial F_{2s}}{\partial G} \right)_0 + \left( \frac{\partial Q_2}{\partial G} \right)_0 \cos 2g + \left( \frac{\partial Q_3}{\partial G} \right)_0 \sin g \right] \\
 & + \left\{ \left( \frac{\partial^2 F_1}{\partial G^2} \right)_0 + \left( \frac{\partial^2 F_{2s}}{\partial G^2} \right)_0 + \left( \frac{\partial^2 Q_2}{\partial G^2} \right)_0 \cos 2g + \left( \frac{\partial^2 Q_3}{\partial G^2} \right)_0 \sin g \right\} \delta \\
 & + \frac{1}{2} \left( \frac{\partial^3 F_1}{\partial G^3} \right)_0 \delta^2 + \frac{1}{6} \left( \frac{\partial^4 F_1}{\partial G^4} \right)_0 \delta^3 .
 \end{aligned} \tag{15}$$

Thus  $\dot{g}$  is a function of  $g$ , and consequently  $g$  may be found for any given time  $t$  by integration:

$$g(t) - g(t_0) = \int_{t_0}^t \dot{g} dt. \tag{16}$$

Now if values of  $g$  and  $\delta$  exist such that  $\dot{g} = 0$ , the motion of  $g$  will be a pure libration. If we designate by  $g_e$  the value of  $g$  satisfying the equation  $\dot{g} = 0$ , the period of libration is given by

$$\frac{T}{2} = \int_{-g_e}^{g_e} \frac{dg}{\dot{g}}. \tag{17}$$

An estimate of the condition for libration is given by

$$\left[ 1 - \frac{1}{4} \left( \frac{Q_3}{Q_2} \right)_0 \right]^2 - (\alpha^2 + \beta^2) > 0, \tag{18}$$

where

$$\alpha^2 = \frac{\left( \frac{\partial^2 F_1}{\partial G^2} \right)_0^2}{\left| 4(Q_2)_0 \left( \frac{\partial^2 F_1}{\partial G^2} \right)_0 \right|},$$

$$\beta^2 = \sin^2 \xi_0 - \frac{(Q_3)_0}{(Q_2)_0} \sin^2 \frac{\xi_0}{2} ,$$

$$\xi_0 = g_0 - \frac{\pi}{2} ,$$

$$\xi = g_e - \frac{\pi}{2} .$$

An estimate of the value of  $g$  satisfying the equation  $\dot{g} = 0$  is given by

$$\sin^2 \frac{\xi}{2} = \frac{1}{2} \left[ 1 - \frac{1}{4} \left( \frac{Q_3}{Q_2} \right)_0 \right] - \frac{1}{2} \sqrt{\left[ 1 - \frac{1}{4} \left( \frac{Q_3}{Q_2} \right)_0 \right]^2 - (\alpha^2 + \beta^2)} , \quad (19)$$

since

$$\frac{1}{4} \left( \frac{Q_3}{Q_2} \right)_0 < 1 .$$

The period of oscillation superimposed on the secular motion when  $\dot{g} \neq 0$  is given by

$$\frac{T}{2} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{dg}{\dot{g}} . \quad (20)$$

#### The Motions of $l$ and $h$

The motions of  $l$  and  $h$  are given by the equations

$$\left. \begin{aligned} l(t) - l(t_0) &= \int_{t_0}^t \dot{l} dt , \\ h(t) - h(t_0) &= \int_{t_0}^t \dot{h} dt \end{aligned} \right\} \quad (21)$$

respectively, where  $\dot{l}$  and  $\dot{h}$  are given by Equation 13; and  $g(t)$  is given in the previous section. The following expressions are provided in order to form  $\dot{l}$  and  $\dot{h}$  from Equation 2:

$$\frac{\partial F_0}{\partial L} = -\frac{1}{L^3}; \quad (22)$$

$$\frac{\partial (F_1)_0}{\partial L} = \frac{3 k_2}{2 L^4 G_0^3} (1 - 3 \cos^2 I_0);$$

$$\begin{aligned} \frac{\partial (F_2)_0}{\partial L} = & \frac{-k_2^2}{32 L^6 G_0^5} \left[ -123 + 1710 \cos^2 I_0 - 2235 \cos^4 I_0 + 48 \frac{L}{G_0} (1 - 6 \cos^2 I_0 + 9 \cos^4 I_0) \right. \\ & \left. + 3 \frac{L^2}{G_0^2} (51 - 630 \cos^2 I_0 + 875 \cos^4 I_0) \right]; \end{aligned}$$

$$\frac{\partial (Q_2)_0}{\partial L} = -\frac{k_2^2}{L^6 G_0^5} \left( \frac{19}{16} - 8 \cos^2 I_0 + \frac{109}{16} \cos^4 I_0 \right) \left( 5 - 3 \frac{L^2}{G_0^2} \right);$$

$$\frac{\partial (Q_3)_0}{\partial L} = \frac{3 k_3 \sin I_0 (1 - 5 \cos^2 I_0) (4e_0^2 - 1)}{4e_0 L^4 G_0^5};$$

$$\frac{\partial}{\partial L} \left( \frac{\partial F_1}{\partial G} \right)_0 = -\frac{3}{L_0} \left( \frac{\partial F_1}{\partial G_0} \right);$$

$$\frac{\partial}{\partial L} \left( \frac{\partial^2 F_1}{\partial G^2} \right)_0 = -\frac{3}{L_0} \left( \frac{\partial^2 F_1}{\partial G^2} \right)_0;$$

$$\frac{\partial}{\partial L} \left( \frac{\partial^3 F_1}{\partial G^3} \right)_0 = -\frac{3}{L_0} \left( \frac{\partial^3 F_1}{\partial G^3} \right)_0;$$

$$\frac{\partial}{\partial L} \left( \frac{\partial^4 F_1}{\partial G^4} \right)_0 = -\frac{3}{L_0} \left( \frac{\partial^4 F_1}{\partial G^4} \right)_0;$$

$$\begin{aligned} \frac{\partial}{\partial L} \left( \frac{\partial F_{2s}}{\partial G} \right)_0 &= - \frac{k_2^2}{L^6 G_0^6} \left[ \frac{615}{32} - \frac{5985}{16} \cos^2 I_0 + \frac{20115}{32} \cos^4 I_0 \right. \\ &\quad \left. - \frac{L}{G_0} (9 - 72 \cos^2 I_0 + 135 \cos^4 I_0) - \frac{L^2}{G_0^2} \left( \frac{1071}{32} - \frac{8505}{16} \cos^2 I_0 + \frac{28875}{32} \cos^4 I_0 \right) \right] ; \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial L} \left( \frac{\partial^2 F_{2s}}{\partial G^2} \right)_0 &= \frac{k_2^2}{L^6 G_0^7} \left[ \frac{1845}{16} - \frac{5985}{2} \cos^2 I_0 + \frac{100575}{16} \cos^4 I_0 \right. \\ &\quad \left. - \frac{L}{G_0} (63 - 648 \cos^2 I_0 + 1485 \cos^4 I_0) - \frac{L^2}{G_0^2} \left( \frac{1071}{4} - \frac{42525}{8} + \frac{86625}{8} \cos^4 I_0 \right) \right] ; \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial L} \left( \frac{\partial Q_2}{\partial G} \right)_0 &= \frac{k_2^2}{L^6 G_0^6} \left[ \frac{475}{16} - 280 \cos^2 I_0 + \frac{4905}{16} \cos^4 I_0 \right. \\ &\quad \left. - \frac{L^2}{G_0^2} \left( \frac{399}{16} - 216 \cos^2 I_0 + \frac{3597}{16} \cos^4 I_0 \right) \right] ; \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial L} \left( \frac{\partial^2 Q_2}{\partial G^2} \right)_0 &= \frac{-k_2^2}{L^6 G_0^7} \left[ \frac{1425}{8} - 2240 \cos^2 I_0 + \frac{24525}{8} \cos^4 I_0 \right. \\ &\quad \left. - \frac{L^2}{G_0^2} \left( \frac{399}{2} - 2160 \cos^2 I_0 + \frac{10791}{4} \cos^4 I_0 \right) \right] ; \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial L} \left( \frac{\partial Q_3}{\partial G} \right)_0 &= \frac{k_3}{e_0^2 L^4 G_0^6} \left\{ \frac{15}{4} e_0 \sin I_0 (1 - 7 \cos^2 I_0) (1 - 4e_0^2) \right. \\ &\quad \left. + \frac{3}{4} (5 \cos^2 I_0 - 1) \left[ \frac{\sin I_0}{e_0} (1 + 2e_0^2) + \frac{e_0}{\sin I_0} (1 - 4e_0^2) \right] \right\} ; \end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial L} \left( \frac{\partial^2 Q_3}{\partial G^2} \right)_0 &= - \frac{3k_3}{4L^4 G_0^7} \left( \frac{10 \sin I_0}{e_0} (3 - 28 \cos^2 I_0) (1 - 4e_0^2) \right. \\
&\quad \left. - \frac{(11 - 75 \cos^2 I_0)}{e_0^2} \left[ \frac{\sin I_0}{e_0} (1 + 2e_0^2) + \frac{e_0}{\sin I_0} (1 - 4e_0^2) \right] \right. \\
&\quad \left. + (5 \cos^2 I_0 - 1) \left\{ \left( \frac{\sin I_0}{e_0} + \frac{e_0}{\sin I_0} \right) \left[ -3 \cot^2 I_0 - \frac{(1 - e_0^2)(2 + 3e_0^2)}{e_0^4} \right] \right. \right. \\
&\quad \left. \left. - \frac{(1 - e_0^2)}{e_0^2} \left( \frac{\sin I_0}{e_0} - \frac{e_0}{\sin I_0} \right) \left( \cot^2 I_0 + \frac{1 - e_0^2}{e_0^2} \right) \right\} \right) ;
\end{aligned}$$

$$\frac{\partial F_0}{\partial H} = 0 ;$$

$$\frac{\partial (F_1)_0}{\partial H} = \frac{3k_2 \cos I_0}{L^3 G_0^4} ;$$

$$\begin{aligned}
\frac{\partial (F_{2s})_0}{\partial H} &= \frac{k_2^2}{8 L^5 G_0^6} \left[ 171 \cos I_0 - 447 \cos^3 I_0 - 36 \frac{L}{G_0} (\cos I_0 - 3 \cos^3 I_0) \right. \\
&\quad \left. - 5 \frac{L^2}{G_0^2} (63 \cos I_0 - 175 \cos^3 I_0) \right] ;
\end{aligned}$$

$$\frac{\partial (Q_2)_0}{\partial H} = \frac{-k_2^2}{4L^5 G_0^6} \left( 1 - \frac{L^2}{G_0^2} \right) (64 \cos I_0 - 109 \cos^3 I_0) ;$$

$$\frac{\partial (Q_3)_0}{\partial H} = \frac{3k_3 e_0 \cot^2 I_0 (11 - 15 \cos^2 I_0)}{4L^3 G_0^6} ;$$

$$\frac{\partial}{\partial H} \left( \frac{\partial F_1}{\partial G} \right)_0 = - \frac{15 k_2 \cos I_0}{L^3 G_0^5} ;$$

$$\frac{\partial}{\partial H} \left( \frac{\partial^2 F_1}{\partial G^2} \right)_0 = \frac{90 k_2 \cos I_0}{L^3 G_0^6} ;$$

$$\frac{\partial}{\partial H} \left( \frac{\partial^3 F_1}{\partial G^3} \right)_0 = \frac{-630 k_2 \cos I_0}{L^3 G_0^7} ;$$

$$\frac{\partial}{\partial H} \left( \frac{\partial^4 F_1}{\partial G^4} \right)_0 = \frac{5040 k_2 \cos I_0}{L^3 G_0^8} ;$$

$$\begin{aligned} \frac{\partial}{\partial H} \left( \frac{\partial F_{2s}}{\partial G} \right)_0 &= \frac{-k_2^2}{L^5 G_0^7} \left[ \frac{1197}{8} \cos I_0 - \frac{4023}{8} \cos^3 I_0 - \frac{L}{G_0} (36 \cos I_0 - 135 \cos^3 I_0) \right. \\ &\quad \left. - \frac{L^2}{G_0^2} \left( \frac{2835}{8} \cos I_0 - \frac{9625}{8} \cos^3 I_0 \right) \right] ; \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial H} \left( \frac{\partial^2 F_{2s}}{\partial G^2} \right)_0 &= \frac{k_2^2}{L^5 G_0^8} \left[ 1197 \cos I_0 - \frac{20115}{4} \cos^3 I_0 - \frac{L}{G_0} (324 \cos I_0 - 1485 \cos^3 I_0) \right. \\ &\quad \left. - \frac{L^2}{G_0^2} \left( \frac{14175}{4} \cos I_0 - \frac{28875}{2} \cos^3 I_0 \right) \right] ; \end{aligned}$$

$$\frac{\partial}{\partial H} \left( \frac{\partial Q_2}{\partial G} \right)_0 = \frac{k_2^2}{L^5 G_0^7} \left[ 112 \cos I_0 - \frac{981}{4} \cos^3 I_0 + \frac{L^2}{G^2} \left( -144 \cos I_0 + \frac{1199}{4} \cos^3 I_0 \right) \right] ;$$

$$\frac{\partial}{\partial H} \left( \frac{\partial^2 Q_2}{\partial G^2} \right)_0 = \frac{-k_2^2}{L^5 G_0^8} \left[ 896 \cos I_0 - \frac{4905}{2} \cos^3 I_0 - \frac{L^2}{G_0^2} (1440 \cos I_0 - 3597 \cos^3 I_0) \right] ;$$

$$\begin{aligned}
\frac{\partial}{\partial H} \left( \frac{\partial Q_3}{\partial G} \right)_0 &= - \frac{3k_3 \cos I_0}{4L^3 G_0^7} \left\{ \frac{5e_0}{\sin I_0} (15 - 21 \cos^2 I_0) \right. \\
&\quad \left. + (1 - 5 \cos^2 I_0) \left( \frac{1}{e_0 \sin I_0} + \frac{e_0}{\sin^3 I_0} \right) + 10 \left( \frac{\sin I_0}{e_0} - \frac{e_0}{\sin I_0} \right) \right\}; \\
\frac{\partial}{\partial H} \left( \frac{\partial^2 Q_3}{\partial G^2} \right)_0 &= \frac{3k_3}{4L^3 G_0^8} \cos I_0 \left\{ 10 \frac{e_0}{\sin I_0} (59 - 84 \cos^2 I_0) + 150 \left( \frac{\sin I_0}{e_0} - \frac{e_0}{\sin I_0} \right) \right. \\
&\quad + (11 - 75 \cos^2 I_0) \left( \frac{1}{e_0 \sin I_0} + \frac{e_0}{\sin^3 I_0} \right) \\
&\quad - 10 \left( \frac{\sin I_0}{e_0} + \frac{e_0}{\sin I_0} \right) \left( \cot^2 I_0 + \frac{1 - e_0^2}{e_0^2} \right) \\
&\quad + (5 \cos^2 I_0 - 1) \left[ \left( \frac{1}{e_0 \sin I_0} - \frac{e_0}{\sin^3 I_0} \right) \left( \cot^2 I_0 + \frac{1 - e_0^2}{e_0^2} \right) \right. \\
&\quad \left. \left. - 2 \left( \frac{1}{e_0 \sin I_0} + \frac{e_0}{\sin^3 I_0} \right) (\cot^2 I_0 + 1) \right] \right\}.
\end{aligned}$$

The Delaunay Variable G

Hagihara (Reference 1) has shown that the Delaunay variable G is bounded. From the canonical equations,

$$\dot{G} = \frac{\partial F}{\partial g}; \quad (23)$$

consequently

$$\begin{aligned}
G &= G_0 + \int_{t_0}^t \left\{ -2 \sin 2g \left[ (Q_2)_0 + \left( \frac{\partial Q_2}{\partial G} \right)_0 \delta + \frac{1}{2} \left( \frac{\partial^2 Q_2}{\partial G^2} \right)_0 \delta^2 \right] \right. \\
&\quad \left. + \cos g \left[ (Q_3)_0 + \left( \frac{\partial Q_3}{\partial G} \right)_0 \delta + \frac{1}{2} \left( \frac{\partial^2 Q_3}{\partial G^2} \right)_0 \delta^2 \right] \right\} dt, \quad (24)
\end{aligned}$$

since  $\delta$  is given as a function of  $g$  by Equations 2 and 3, and  $g$  is given as a function of  $t$  as discussed above,  $G$  can be evaluated for any time  $t$ .

## Procedure for Computations

The computation proceeds along the following lines:

1. The mean motion of the variables  $l$ ,  $g$ , and  $h$  are found;
2. The periods of libration or oscillation are computed;
3. The angular variable  $g$  is found as a function of time from Equations 14-20; and
4. The variable  $G$  as well as the motions of the remaining variables are then found.

## REDUCTION TO THE EXTENDED PENDULUM PROBLEM

It is interesting to show how the aforementioned may be approximated by an extended pendulum problem, and consequently, how the estimates for libration were derived. An approximation to Equation 6 is given by

$$(Q_2)_0 [\cos 2g - \cos 2g_0] + (Q_3)_0 [\sin g - \sin g_0] + \left(\frac{\partial F_1}{\partial G}\right)_0 \delta + \frac{1}{2} \left(\frac{\partial^2 F_1}{\partial G^2}\right)_0 \delta^2 = 0. \quad (25)$$

Then by setting

$$\xi = g - \frac{\pi}{2},$$

and

$$\xi_0 = g_0 - \frac{\pi}{2}$$

we can put Equation 25 into the form

$$\frac{\left(\frac{\partial F_1}{\partial G}\right)_0}{2(Q_2)_0} \delta + \frac{\left(\frac{\partial^2 F_1}{\partial G^2}\right)_0}{4(Q_2)_0} \delta^2 + \sin^2 \xi - \left(\frac{Q_3}{Q_2}\right)_0 \sin^2 \frac{\xi}{2} = \beta^2, \quad (26)$$

where

$$\beta^2 = \sin^2 \xi_0 - \left(\frac{Q_3}{Q_2}\right)_0 \sin^2 \frac{\xi_0}{2}.$$

Following Garfinkel (Reference 4), we set

$$\eta = \delta \left[ \frac{\left( \frac{\partial^2 F_1}{\partial G^2} \right)_0}{(4Q_2)_0} \right]^{1/2} + \alpha ,$$

where

$$\alpha = \frac{\left( \frac{\partial^2 F_1}{\partial G^2} \right)_0}{\left[ 4(Q_2)_0 \left( \frac{\partial^2 F_1}{\partial G^2} \right)_0 \right]^{1/2}} .$$

Now Equation 26 takes the form

$$\eta^2 + \sin^2 \xi - \left( \frac{Q_3}{Q_2} \right)_0 \sin^2 \frac{\xi}{2} = \alpha^2 + \beta^2 . \quad (27)$$

With

$$\dot{\delta} = \frac{\partial F}{\partial g} = \frac{\partial F}{\partial \xi} ,$$

we find that

$$\dot{\delta} = 2(Q_2)_0 \sin 2\xi - (Q_3)_0 \sin \xi . \quad (28)$$

Using the definition of  $\eta$  given by Equations 26 and 28, we readily find that

$$\dot{\eta} = 2(Q_2)_0 \left[ \frac{\left( \frac{\partial^2 F_1}{\partial G^2} \right)_0}{4(Q_2)_0} \right]^{1/2} \left[ \sin 2\xi - \frac{(Q_3)_0}{2(Q_2)_0} \sin \xi \right] . \quad (29)$$

Also, by differentiating Equation 27, and using Equation 29 we find that

$$\eta = - \frac{\dot{\xi}}{\left[ 4(Q_2)_0 \left( \frac{\partial^2 F_1}{\partial G^2} \right)_0 \right]^{1/2}} . \quad (30)$$

By substituting Equation 30 into Equation 27, an extended form of the pendulum problem is

$$\frac{\dot{\xi}^2}{4(Q_2)_0 \left( \frac{\partial^2 F_1}{\partial G^2} \right)_0} = \alpha^2 + \beta^2 - \sin^2 \xi + \frac{(Q_3)_0}{(Q_2)_0} \sin^2 \frac{\xi}{2} . \quad (31)$$

To find the conditions for libration we set  $y = \sin \xi/2$  in the right side of Equation 31 to obtain

$$\frac{\dot{\xi}^2}{4(Q_2)_0 \left( \frac{\partial^2 F_1}{\partial Q^2} \right)_0} = (\alpha^2 + \beta^2) - 4y^2(1 - y^2) + \left( \frac{Q_3}{Q_2} \right)_0 y^2 .$$

The libration condition  $\dot{\xi} = 0$  is then expressed by the condition that

$$y^4 - y^2 \left[ 1 - \frac{1}{4} \left( \frac{Q_3}{Q_2} \right)_0 \right] + \frac{(\alpha^2 + \beta^2)}{4} = 0 .$$

Hence, for a solution to exist (i.e., for libration to occur), we must have

$$\sin^2 \frac{\xi}{2} = \frac{1}{2} \left[ 1 - \frac{1}{4} \left( \frac{Q_3}{Q_2} \right)_0 \right] - \frac{1}{2} \sqrt{\left[ 1 - \frac{1}{4} \left( \frac{Q_3}{Q_2} \right)_0 \right]^2 - (\alpha^2 + \beta^2)} , \quad (32)$$

which means that

$$\left[ 1 - \frac{1}{4} \left( \frac{Q_3}{Q_2} \right)_0 \right]^2 - (\alpha^2 + \beta^2) > 0 . \quad (33)$$

Equation 32 also yields the following estimate for the libration angle

$$\sin^2 \frac{\xi}{2} = \frac{1}{4} \frac{(\alpha^2 + \beta^2)}{\left[ 1 - \frac{1}{4} \left( \frac{Q_3}{Q_2} \right)_0 \right]}, \quad (34)$$

where, of course,

$$\alpha^2 + \beta^2 \ll 1,$$

$$\frac{1}{4} \frac{(Q_3)_0}{(Q_2)_0} \ll 1.$$

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